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DIFFUSION INDUCED UNSTEADY BOUNDARY FLOWS IN A WEDGE-SHAPED TROUGH†

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The plane problem of the formation stratification in a wedge-shaped trough filled with a homogeneous incompressible liquid is considered when a constantly acting source of admixture is inserted in the bottom of the trough. The low intensity of the induced diffusive and convective flows which arise enables one to use the methods of perturbation theory. Application of the method of integral transforms to the resulting linearized systems of equations gives the solution for the velocity and admixture fields in the form of quadrature formulae, on the basis of which the characteristic quantities and scales of the flows which arise can be determined. © 1999 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

The processes of the formation of stratification in a trough are considered. The trough is formed by two semi-infinite inclined impermeable walls and is filled with a homogeneous incompressible liquid. At the initial instant of time, a source of salt at the bottom, in the form of an arc of a circle of radius **a** with an aperture angle 2α (Fig. 1), which is characterized by a coefficient of salt release γ_s and a value S_{00} of the salt concentration on its surface, is released.

The unsteady problem of determining the velocity and salinity fields which will be formed in the trough is considered in a two-dimensional formulation. In a polar system of coordinates, shown in Fig. 1, the boundary conditions for the source are written in the form

$$\left\{k_s\frac{\partial S}{\partial r}+\gamma_s(S-S_{00})\right\}_{r=a}=0 \tag{1.1}$$

where S is the required distribution of the salt in the trough and k_s is the diffusion coefficient of the salt. The no-slip condition for the velocity is satisfied on the bottom and on the inclined walls. Furthermore, the impermeability condition for the salt is satisfied on the inclined walls. The flow develops in the field of the gravitation force **g**. All perturbations decay at infinity.

After the source of salt has been released, the hydrodynamic equilibrium of the liquid is destroyed. This is due to the fact that, in the first stages of the development of the flow, the diffusion mechanism forms a density distribution with isohalines (equisalinity lines) in the form of concentric circles with their centre at the origin of the polar system of coordinates ρ, φ. At the same time, only density distributions with horizontal isohalines are stable in a gravitation force field. As a result of this, a flow rises under the action of an Archimedean force which tends to "straighten out" the isohalines.

If there were no inclined impermeable walls, the isohalines at a sufficient distance from the source would take the form of a horizontal planes since, in the majority of practical situations, the characteristic times of inertial processes under the action of the gravity force are much shorter than the characteristic times of diffusion processes. However, the presence of the impermeable inclined walls does, in fact, change the nature of the flow considerably since the condition that there is no flux of salt on the inclined walls leads to a state of affairs where, close to the surface of these walls, the lines of equisalinity are always normal to them. As a result of this, a mechanism for the generation of a flow close to the wall, which compensates for the inclination of isohalines to the horizontal, and for the generation of a descending flow in the central domain of the trough which closes a vortex pattern, is constantly present in the flow.

In writing down the equations of fluid dynamics, condition (1.1) enables us to avoid having to introduce a source term and to use the system of standard equations in the Boussinesq approximation

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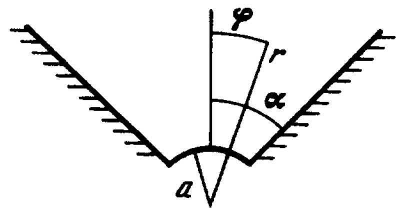

Fig. 1.

$$\partial \mathbf{u}/\partial t + (\mathbf{u}\nabla)\mathbf{u} = -\nabla \mathbf{p} + \nu\Delta\mathbf{u} + \mathbf{g}S, \quad \nabla\cdot\mathbf{u} = 0$$

$$\partial S/\partial t + \mathbf{u}\cdot\nabla S = k_s\Delta S \tag{1.2}$$

Here, **u** is the velocity field and p is the pressure, after subtracting the hydrostatic pressure, normalized to ρ_0, the density of the homogeneous liquid at the instant when the source is released. In (1.1) and (1.2), the salinity is represented by the coefficient of salt contraction and is therefore a dimensionless quantity. In this case, the density of the medium is expressed in terms of the salinity using the linearized equation of state with constant coefficients $\rho = \rho_0(1 + \beta S)$, where β is the coefficient of salt contraction.

2. CHOICE OF THE FORM OF SOLUTION

We will seek the solution of problem (1.1), (1.2) in the form of Fourier series in the angle φ (the summation is from $n = 1$ to $n = \infty$)

$$u = \sum u_n(r,t)\cos\frac{n\pi\varphi}{\alpha}, \quad p = p_0(r,t) + \sum p_n(r,t)\cos\frac{n\pi\varphi}{\alpha}$$

$$v = \sum v_n(r,t)\sin\frac{n\pi\varphi}{\alpha}, \quad S = S_0(r,t) + \sum S_n(r,t)\cos\frac{n\pi\varphi}{\alpha} \tag{2.1}$$

where u, υ are the radial and azimuthal components of the velocity.

Representation (2.1) guarantees that the boundary conditions on the inclined walls are satisfied.

Due to the slowness of the transient process and the low intensity of the flows induced by transport phenomena close to the inclined walls, the problem is solved using the method of successive approximations, in which the Fourier coefficients in (2.1) are expanded in series in powers of the smallness of the effect of the non-linear terms of Eqs (1.2)

$$S_n(r,t) = S_n^{(0)} + S_n^{(1)} + S_n^{(2)} + \ldots \tag{2.2}$$

Similar series also hold for u_n, υ_n and p_n.

By substituting (2.2) into (1.1), (1.2) we can obtain a set of successive systems of equations and boundary conditions for $S_n^{(i)}$, $u_n^{(i)}$, $\upsilon_n^{(i)}$ and $p_n^{(i)}$ where the general conditions at infinity and the initial conditions

$$u_n^{(i)} = v_n^{(i)} = S_n^{(i)} = p_n^{(i)} = 0 \text{ when } r \to \infty \text{ and when } t = 0$$

are satisfied for all i.

Furthermore, the additional boundary condition

$$\partial u_n^{(i)}/\partial r\,|_{r=a} = 0$$

follows from the second equation of (1.2).

For the terms of the zeroth approximation, a system of equations of the form

$$\frac{\partial S_n^{(0)}}{\partial t} = k_s\left(\frac{\partial^2 S_n^{(0)}}{\partial^2 r} + \frac{1}{r}\frac{\partial S_n^{(0)}}{\partial r} - \left(\frac{n\pi}{\alpha}\right)^2\frac{S_n^{(0)}}{r^2}\right), \quad n = 0, 1, \ldots \tag{2.3}$$

is obtained with the initial and boundary conditions

$$S_n^{(0)}(r,0)=0,\quad S_n^{(0)}(\infty,t)=0$$

$$\left(k_s\frac{\partial S_0^{(0)}}{\partial r}+\gamma_s S_0^{(0)}\right)\Bigg|_{r=a}=\gamma_s S_{00},\quad \left(k_s\frac{\partial S_n^{(0)}}{\partial r}+\gamma_s S_n^{(0)}\right)\Bigg|_{r=a}=0$$

It is clear that $u_n^{(0)}=\upsilon_n^{(0)}=S_n^{(0)}=0$ for $n=1,2,\ldots$.

3. CONSTRUCTION OF THE SOLUTION IN THE FORM OF QUADRATURES USING INTEGRAL TRANSFORMS

The solution of Eq. (2.3) is found by the method of integral transforms. We introduce the function

$$G_n^{(0)}(\xi,t)=\int_a^\infty S_n^{(0)}(r,t)\,\mathcal{L}(r,\xi,t)\rho(r)dr$$

where $\mathcal{L}(r,\xi,t)$ is the kernel of the integral transform and $\rho(r)$ is a weighting function.

On applying the integral transform, which defines the function $G_n^{(0)}(\xi,t)$, to system (2.3) and imposing the condition that the transformed equation should not contain integral terms, we obtain the relations

$$\rho=r,\quad \frac{\partial^2\mathcal{L}_n}{\partial r^2}+\frac{1}{r}\frac{\partial\mathcal{L}_n}{\partial r}+\left(\xi^2-\frac{1}{r^2}\left(\frac{n\pi}{\alpha}\right)^2\right)\mathcal{L}_n=0,\quad n=0,1,2,\ldots$$

$$\frac{\partial G_n^{(0)}}{\partial t}=-k_s\xi^2G_n^{(0)}+k_s\left(\frac{\partial S_n^{(0)}}{\partial r}\mathcal{L}_n r-rS_n^{(0)}\frac{\partial\mathcal{L}_n^{(0)}}{\partial r}\right)\Bigg|_a^\infty$$

whence, using the initial and boundary conditions, we obtain

$$G_n^{(0)}=0,\quad n=1,2,\ldots$$

$$G_0^{(0)}(\xi,t)=a\frac{\gamma_s S_{00}^{(0)}}{k_s\xi^2}\mathcal{L}_0^{(0)}(a,\xi)(\exp(-k_s\xi^2t)-1)\tag{3.1}$$

Here, the kernel $\mathcal{L}_0^{(0)}$ is the solution of the boundary-value problem

$$\frac{\partial^2\mathcal{L}_0^{(0)}}{\partial r^2}+\frac{1}{r}\frac{\partial\mathcal{L}_0^{(0)}}{\partial r}+\xi^2\mathcal{L}_0^{(0)}=0\quad \left(k_s\frac{\partial\mathcal{L}_0^{(0)}}{\partial r}+\gamma_s\mathcal{L}_0^{(0)}\right)\Bigg|_{r=a}=0\tag{3.2}$$

Applying to the solution of Eq. (3.2) to the normalization condition

$$\left(\sqrt{r}\frac{\partial\mathcal{L}_0^{(0)}}{\partial r}\right)\Bigg|_{r=a}=1$$

we finally obtain

$$\mathcal{L}_0^{(0)}=\frac{\pi}{2\sqrt{a}}\operatorname{sign}\left(\frac{1}{2}-\frac{a\gamma_s}{k_s}\right)\left(a\xi k_1-\frac{a\gamma_s}{k_s}k_2\right)\left(1+\left(\frac{1}{2}-\frac{a\gamma_s}{k_s}\right)^2\right)^{-1/2}\tag{3.3}$$

$$k_1=J_1(a\xi)N_0(r\xi)-J_0(r\xi)N_1(a\xi),\quad k_2=J_0(\alpha\xi)N_0(r\xi)-J_0(r\xi)N_0(a\xi)$$

where J and N are Bessel Neuman functions.

Now, having the solutions for the function $G_0^{(0)}(\xi,t)$, the form of the kernel $K_0^{(0)}$ and the weighting function ρ, it merely remains to invert the integral transform in order to obtain the solution $S_0^{(0)}(r,t)$.

It is well known [2] that, if

$$g(\xi)=\int_a^\infty \psi(r,\xi)g(r)dr,\quad g(r)\in \mathcal{L}^2(a,\infty)$$

where $\psi(r,\xi)$ is the solution of the corresponding initial boundary-value problem, the following relation holds

$$g(r)=\int_{-\infty}^{\infty} g(\xi)\psi(r,\xi)d\sigma(\xi)$$

in which the last integral is defined in the Lebesgue–Stieltjes sense.

The differential of the spectral function $\sigma(\xi)$ is determined by taking the limit

$$d\sigma(\xi)=\frac{1}{\pi}\lim_{\varepsilon\to+0}\operatorname{Im} m_\infty(\xi+i\varepsilon)d\xi$$

where m_∞ is the equation of the limiting perimeter which bounds the domain of the solutions of class $\mathcal{L}^2(a,\infty)$ of the singular Sturm–Liouville problem.

Omitting the intermediate operations involved in calculating the differential of the spectral function $\sigma(\xi)$, we now present the final result for the inverse transform

$$S_0^{(0)}(r,t)=A_0\int_0^\infty K_0^{(0)}(r,\xi)(e^{-k_s\xi^2 t}-1)d\xi,\quad A_0=\frac{2}{\pi}\frac{a\gamma_s}{k_s}S_{00} \tag{3.4}$$

$$K_0^{(0)}=\frac{1}{\xi}\left(a\xi k_1-\frac{a\gamma_s}{k_s}k_2\right)\left\{\left(a\xi J_1(a\xi)-\frac{a\gamma_s}{k_s}J_0(a\xi)\right)^2+\left(a\xi N_1(a\xi)-\frac{a\gamma_s}{k_s}N_0(a\xi)\right)^2\right\}^{-1}$$

This approximation describes the initial stage of the formation of the flow when the salinity distribution is exclusively created by diffusion of the particles in the liquid and depends solely on the radial coordinate r and there is no motion in the medium. As a result of this, a salinity distribution is formed with isohalines which are arcs of circles with centre at the origin of coordinates, the hydrostatic equilibrium of the liquid is destroyed and a flow arises which compensates for both the deficit of salinity close to the inclined walls of the trough as well as the excess salinity at the centre of the trough.

After the zeroth approximation of the salinity distribution, $S^{(0)}$, has been calculated, we now determine which velocity field arises in the medium as a reaction to this zeroth approximation. The system of equations for the first approximation of the velocity field with the initial and boundary conditions

$$u_n^{(1)}(r,0)=\upsilon_n^{(1)}(r,0)=0,\quad u_n^{(1)}(a,t)=\upsilon_n^{(1)}(a,t)=\left.\frac{\partial u_n^{(1)}(r,t)}{\partial t}\right|_{r=a}=0$$

has the form

$$\frac{\partial}{\partial r}(ru_n^{(1)})+\frac{n\pi}{\alpha}\upsilon_n^{(1)}=0,\quad n=1,2,\ldots$$

$$\frac{\partial u_n^{(1)}}{\partial t}=-\frac{\partial p_n^{(1)}}{\partial r}+\nu\left(\frac{\partial^2 u_n^{(1)}}{\partial r^2}+\frac{1}{r}\frac{\partial u_n^{(1)}}{\partial r}-\frac{1}{r^2}\left(\left(1+\left(\frac{n\pi}{\alpha}\right)^2\right)u_n^{(1)}+2\frac{n\pi}{\alpha}\upsilon_n^{(1)}\right)\right)+$$
$$+2S_0^{(0)}g(-1)^n\frac{\alpha\sin\alpha}{(n\pi)^2-\alpha^2} \tag{3.5}$$

$$\frac{\partial \upsilon_n^{(1)}}{\partial t}=\frac{n\pi}{\alpha r}p_n^{(0)}+\nu\left(\frac{\partial^2 \upsilon_n^{(1)}}{\partial r^2}+\frac{1}{r}\frac{\partial \upsilon_n^{(1)}}{\partial r}-\frac{1}{r^2}\left(\left(1+\left(\frac{n\pi}{\alpha}\right)^2\right)\upsilon_n^{(1)}+2\frac{n\pi}{\alpha}u_n^{(1)}\right)\right)-$$
$$-2S_0^{(0)}g(-1)^n\frac{n\pi\sin\alpha}{(n\pi)^2-\alpha^2}$$

where u, υ are the radial and azimuthal components of the velocity, respectively.

Solving problem (3.5) using the method of integral transforms in the same way as above in the case of $S_0^{(0)}$, we obtain quadrature formulae for the first approximation of the velocity field

$$u_n^{(1)}(r,t) = -A_n\left(\frac{n\pi}{\alpha}\right)^2 \int_0^\infty \frac{\Phi_n(\xi,t)L_n(r,\xi)}{\xi\{J_{\mu_n}^2(a\xi)+N_{\mu_n}^2(a\xi)\}}d\xi$$

$$v_n^{(1)}(r,t) = -\frac{\alpha}{n\pi}\frac{\partial}{\partial r}(ru_n^{(1)}) \tag{3.6}$$

$$L_n = \frac{1}{r\sqrt{a}}(N_{\mu_n}(r\xi)J_{\mu_n}(a\xi) - J_{\mu_n}(r\xi)N_{\mu_n}(a\xi)), \quad \mu_n = \frac{n\pi}{\alpha}$$

$$\Phi_n = \int_a^\infty r^2 L_n \frac{\partial}{\partial r}\left\{\int_0^\infty K_0^{(0)}\left(\frac{e^{-k_s\eta^2 t} - e^{-\nu\xi^2 t}}{\nu\xi^2 - k_s\eta^2} + \frac{e^{-\nu\xi^2 t}-1}{\nu\xi^2}\right)d\eta\right\}dr$$

$$A_n = 2g(-1)^n\frac{\alpha\sin\alpha}{(n\pi)^2-\alpha^2}A_0$$

The flow which has arisen distorts the zeroth approximation distribution $S_0^{(0)}$ and leads to the appearance of the corrections of the first approximation which are determined as the solution of a initial-boundary-value problem of the form

$$\frac{\partial S_0^{(1)}}{\partial t} + \frac{1}{2r}\sum_{n=1}^\infty \frac{\partial}{\partial r}(rS_n^{(0)}u_n^{(1)}) = k_s\left(\frac{\partial^2 S_0^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial S_0^{(1)}}{\partial r}\right)$$

$$\frac{\partial S_n^{(1)}}{\partial t} - k_s\left(\frac{\partial^2 S_0^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial S_0^{(1)}}{\partial r} - \left(\frac{n\pi}{\alpha}\right)^2\frac{S_n^{(1)}}{r^2}\right) = -u_n^{(1)}\frac{\partial S_0^{(0)}}{\partial r}, \quad n = 1,2,\dots \tag{3.7}$$

$$k_s\frac{\partial S_n^{(1)}}{\partial r} + \gamma_s S_n^{(1)}\Big|_{r=a} = 0, \quad S_n^{(1)}(r,0) = 0, \quad n = 0,1,2,\dots$$

It is clear from the first approximation that $S_0^{(1)} = 0$ since $S_n^{(0)} = 0$ when $n = 1, 2, \dots$.

The equations for $S_0^{(m)}$ will subsequently have a form which is analogous to the equations for $S_0^{(1)}$ from system (3.7) and, hence, $S_0^{(m)} = 0, m = 1, 2, \dots$ and $S_{00}^{(0)} = S_{00}$.

On solving the remaining equations of (3.7) using the method of integral transforms mentioned above, we obtain the first approximation corrections to the salinity distribution

$$S_n^{(1)} = \frac{2\cos\alpha}{\pi\sqrt{a}}\int_0^\infty K_n^{(1)}(r,\xi)G_n^{(1)}(r,\xi)d\xi$$

$$K_n^{(1)} = \xi\left(a\xi m_1 + \frac{a\gamma_s}{k_s}m_2\right)((a\xi J'_{\mu_n}(a\xi)h + J_{\mu_n}(a\xi)f)^2 + (a\xi N'_{\mu_n}(a\xi)h + N_{\mu_n}(a\xi)f)^2)^{-1} \tag{3.8}$$

$$m_1 = J'_{\mu_n}(a\xi)N_{\mu_n}(r\xi) - N'_{\mu_n}(a\xi)J_{\mu_n}(r\xi), \quad m_2 = J_{\mu_n}(a\xi)N_{\mu_n}(r\xi) - N_{\mu_n}(a\xi)J_{\mu_n}(r\xi)$$

$$h(\alpha) = \cos\alpha, \quad f(\alpha) = \frac{\cos\alpha}{2} - \sin\alpha$$

The function $G_n^{(1)}(\xi, t)$ is the solution of the problem

$$\frac{\partial G_n^{(1)}}{\partial t} + k_s\xi^2 G_n^{(1)} = \int_a^\infty u_n^{(1)}K_n^{(1)}r^3\frac{\partial S_0^{(0)}}{\partial r}dr, \quad G_n^{(1)}(\xi,0) = 0$$

On continuing this process, the corrections of the following approximations can be obtained, the contribution of which to the overall velocity field, salinity field and pressure field distributions will decrease.

4. ANALYSIS OF THE RESULTS

Certain qualitative conclusions can be drawn from the above results. The function $G_n^{(1)}(r, \xi)$, the characteristics of which, in their turn, are specified by the functions $u_n^{(1)}$ and $S_0^{(0)}$, occurs in relation (3.8) which determine the first approximation corrections $S_n^{(1)}$ to the salinity distribution. The quantity $S_0^{(0)}$ is defined by relation (3.4) and the characteristic singularities of the velocity field $u_n^{(1)}$ are described by the integral representation of the function $\Phi_n(\xi, t)$ which occurs in relation (3.6). It is clear from (3.4), (3.6) and (3.8) that the integral representation for $S_n^{(1)}$ is singularly perturbed with respect to the parameter $\mathrm{Sc} = \sqrt{(k_s/\nu)}$ which takes small values in real media (when Sc = 0 the order of the differential equations is reduced). This indicates that, in the present problem, it is impossible to neglect the effects of diffusion compared with viscous effects.

The terms $\exp^{-\nu\eta^2 t}$, $\exp^{-k_s\xi^2 t}$ and $\exp^{-k_s\xi^2 t-\nu\eta^2 t}$ occur in the kernels of the integral expressions (3.4), (3.6) and (3.8). In fact, they determine the formation of boundary flows both close to the walls as well as on the surface of the salt source. In the same way as described previously [3], terms of the first two types describe the formation of velocity and density boundary layers, the thickness of which at the initial instants of time increases as $\delta_u \sim \sqrt{(\nu t)}$ and $\delta_\rho \sim \sqrt{(k_s t)}$. Meanwhile, unlike the results which have been obtained previously [3], the thickness of the layers in a trough of finite depth cannot increase without limit. A term of the last type characterizes the combination scale $\delta_{\rm comb} = (k_s\nu t^2)1/4$, that is, unlike a flow close to an inclined plane, here there is only partial splitting of the scales.

The solutions obtained are analytic with respect to all the physical variables, including γ_s in the sense that, when $\gamma_s \to \infty$, all the solutions pass smoothly into the solutions of a problem in which the condition $S(a, t) = S_{00}$ is used as the boundary condition on the source. Solutions for such boundary conditions are also obtained in the form of quadratures, but they are not given here. It should also be noted that the solutions are also analytic in the limiting cases when the "aperture" angle of the trough is $\alpha = 0$ or $\alpha = \pi/2$.

The use of asymptotic methods to evaluate the integrals for the case when $\xi a \gg 1$ enables one to obtain approximate results which describe the dynamics of the formation of the salinity distribution in the trough.

For example, for the zeroth approximation, we obtain

$$S_0^{(0)}(\rho,t) = S_{00}[\operatorname{erfc}\zeta - 2e^{-\lambda(\rho-1)} + e^{\tau_s\lambda^2}e^{-\lambda(\rho-1)}\operatorname{erfc}(\lambda\sqrt{\tau_s} - \zeta)] \tag{4.1}$$

$$\xi = \frac{\rho-1}{2\sqrt{\tau_s}}, \quad \rho = \frac{r}{a}, \quad \tau_s = \frac{k_s t}{a^2}, \quad \lambda = \frac{a\gamma_s}{k_s}$$

It is clear that this distribution is characterized not only by a boundary layer (the first term in the square brackets) but also by an injection front (the last two terms).

The extremely long expressions for the other asymptotic formulae are not presented here.

It should be noted that there are two new variables ζ and $\lambda\sqrt{(\tau_s)} - \zeta$, obtained on the basis of the integral transforms, in the asymptotic solution (4.1). On the other hand, group-theory analysis of the initial system of equations (1.2) separates out these variables as invariants of the generators of the Lie group corresponding to Eqs (1.2). This fact indicates that it is possible to construct a more universal method for solving such problems which could simplify the derivation of the final formulae (although these would also be quadrature formulae). The essence of such an approach can be formulated as follows. In the first stage, a Lie group analysis of the boundary-value problem under investigation is carried out, the characteristic variables which are convenient for finding the solution are determined, and the problem is then formulated in these new variables. In the second stage, the method of integral transforms is applied to the reformulated problem, the execution of which is considerably facilitated since the variables corresponding to the boundary layers will be separated from the variables describing the structure of the injection fronts.

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